Chapter 10

Relativity

	page
Special relativity	246
Lorentz transformation	246
4-Vector transformation	246
Examples of 4-vectors	247
2nd rank tensor transformation	247
Electromagnetic field strength tensor	247
Covariant formulation of Maxwell's equations	247
Lorentz force	247
Cosmology	248
Robertson–Walker line element	248
Einstein field equations	248
Friedmann universes	248
Bibliography	254

© Cambridge University Press • Provided by the NASA Astrophysics Data System

Special relativity

Fundamental kinematical relations for a particle of rest mass m_0 and velocity v:

$p = m_0 v/(1 - v^2/c^2)^{1/2}$	momentum
$E = m_0 c^2/(1 - v^2/c^2)^{1/2}$	total energy
$T = E - m_0 c^2$	kinetic energy
$m = m_0/(1 - v^2/c^2)^{1/2}$	relativistic mass
$E_0 = m_0 c^2$	rest energy

From the above, the following relations can be derived:

$$E = mc^2 = (m_0^2 c^4 + c^2 p^2)^{1/2}$$

$$p = [(E/c)^2 - m_0^2 c^2]^{1/2}$$

$$v = c^2 p/E = c[1 - (m_0 c^2/E)^2]^{1/2} = p/[m_0 c^2 + (p/c)^2]^{1/2}$$

$$m = E/c^2 = [m_0^2 + (p/c)^2]^{1/2}$$

Relativistic Doppler effect:

$$1 + z = \frac{1 + (v/c)\cos\theta}{(1 - v^2/c^2)^{1/2}},$$

where

θ = angle between direction of observation and direction of motion,
$\theta = 0$ for motion directly away from observer,
$z = (\lambda_{\text{obs}} - \lambda)/\lambda$.

$$z \approx (v/c)\cos\theta \quad \text{for } v << c.$$

Lorentz transformation (Gaussian units)

4-Vector transformation

$$B'_\mu = \sum_{\nu=1}^{4} a_{\mu\nu} B_\nu \equiv a_{\mu\nu} B_\nu .$$

For a Lorentz transformation from system k to a system k' moving with a velocity parallel to the z-axis, the transformation coefficients are given by:

$$(a_{\mu\nu}) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \gamma & i\gamma\beta \\ 0 & 0 & -i\gamma\beta & \gamma \end{bmatrix}, \quad \beta = v/c,\ \gamma = (1 - v^2/c^2)^{-1/2}.$$

© Cambridge University Press • Provided by the NASA Astrophysics Data System

Examples of 4-vectors

$$x_\mu = (\mathbf{x}, ict),$$

where $\mathbf{x} = x_1\hat{i} + x_2\hat{j} + x_3\hat{k}$.

$$A_\mu = (\mathbf{A}, i\phi),$$

where $\mathbf{A}$ and ϕ are the electromagnetic vector and scalar potential.

$$J_\mu = (\mathbf{J}, ic\rho),$$

where $\mathbf{J}$ and ρ are the current density and charge density.

$$k_\mu = (\mathbf{k}, i\omega/c),$$

where $\mathbf{k}$ and ω are the wave vector and frequency of a plane electromagnetic wave.

$$p_\mu = (\mathbf{p}, iE/c),$$

where $\mathbf{p}$ and E are the momentum and energy of a particle.

2nd rank tensor transformation

$$S_{\mu\nu} = \sum_{\lambda,\sigma=1}^{4} a_{\mu\lambda} a_{\nu\sigma} S_{\lambda\sigma} \equiv a_{\mu\lambda} a_{\nu\sigma} S_{\lambda\sigma}.$$

Electromagnetic field strength tensor

$$F_{\mu\nu} = \frac{\partial A_\nu}{\partial x_\mu} - \frac{\partial A_\mu}{\partial x_\nu} = \begin{bmatrix} 0 & B_3 & -B_2 & -iE_1 \\ -B_3 & 0 & B_1 & -iE_2 \\ B_2 & -B_1 & 0 & -iE_3 \\ iE_1 & iE_2 & iE_3 & 0 \end{bmatrix}.$$

Covariant formulation of Maxwell's equations

$$\frac{\partial F_{\mu\nu}}{\partial x_\nu} = \frac{4\pi}{c} J_\mu; \quad \frac{\partial F_{\mu\nu}}{\partial x_\lambda} + \frac{\partial F_{\lambda\mu}}{\partial x_\nu} + \frac{\partial F_{\nu\lambda}}{\partial x_\mu} = 0,$$

where λ, μ, and ν are any three of the integers $1, 2, 3, 4$.

Lorentz force

$$\mathbf{f} = \rho\mathbf{E} + \frac{1}{c}(\mathbf{J} \times \mathbf{B}).$$

$$f_\mu = \frac{1}{c} F_{\mu\nu} J_\nu = \frac{1}{4\pi} F_{\mu\nu} \frac{\partial F_{\nu\lambda}}{\partial x_\lambda} = \frac{\partial T_{\mu\nu}}{\partial x_\nu} = \left(\mathbf{f}, \frac{i}{c}\mathbf{E}\cdot\mathbf{J}\right),$$

where $T_{\mu\nu}$ is the electromagnetic stress–energy–momentum tensor:

$$T_{\mu\nu} = \frac{1}{4\pi}\left[F_{\mu\lambda}F_{\lambda\nu} + \tfrac{1}{4}\delta_{\mu\nu}F_{\lambda\sigma}F_{\lambda\sigma}\right].$$

© Cambridge University Press • Provided by the NASA Astrophysics Data System

Robertson–Walker line element (homogeneous and isotropic universe)

$$ds^2 = c^2\,dt^2 - R^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta\,d\phi^2)\right],$$

where

$R(t)$ = radius of curvature of the universe,

r, θ, ϕ = co-moving spherical coordinates (a co-moving observer is an observer at rest with respect to matter in his vicinity),

k = curvature index $= 0, \pm 1$ ($k = 1$, elliptical closed space; $k = 0$, Euclidean flat space; $k = -1$, hyperbolic open space).

Einstein field equations

$$\frac{3\dot{R}^2}{R^2} + \frac{3kc^2}{R^2} = 8\pi G\rho + \Lambda c^2,$$

$$\frac{2\ddot{R}}{R} + \frac{\dot{R}^2}{R^2} + \frac{kc^2}{R^2} = -\frac{8\pi GP}{c^2} + \Lambda c^2,$$

$$\frac{\ddot{R}}{R} = \frac{\Lambda c^2}{3} - \frac{4\pi G}{3}\left[\rho + \frac{3P}{c^2}\right],$$

where

ρ = mean density of matter and energy,

Λ = cosmological constant,

P = hydrodynamic pressure of matter and radiation,

G = gravitational constant,

$H_0 \equiv \dot{R}_0/R_0$, Hubble constant,

$q_0 \equiv -\ddot{R}_0/R_0H_0^2$, deceleration constant, where the subscript zero denotes the present value.

Friedmann universes ($\Lambda = 0$)

$$\rho_0 + \frac{3P_0}{c^2} = \frac{3H_0q_0}{4\pi G},$$

$$\frac{kc^2}{R_0^2} = \frac{4\pi G}{3q_0}\left[\rho_0(2q_0 - 1) - \frac{3P_0}{c^2}\right].$$

© Cambridge University Press • Provided by the NASA Astrophysics Data System

For epochs after the decoupling of matter, P_0 is approximately 0, and we have

Curvature	Space	q_0	Ω_0[a]	Density ρ_0	Expansion
$k=+1$	Closed	$>1/2$	>1	$>\dfrac{3H_0^2}{8\pi G}$	Turns eventually into contraction
$k=0$	Flat (Euclidean)	$1/2$	1	$\rho_c=\dfrac{3H_0^2}{8\pi G}$ [b]	Stops in infinite future
$k=-1$	Open	$0\leqslant q_0<1/2$	$0\leqslant\Omega_0<1$	$<\dfrac{3H_0^2}{8\pi G}$	Forever

[a] The 'density parameter' $\Omega\equiv\rho/\rho_c$, where ρ_c is the critical closure density (i.e. for the case $q_0=1/2$).

[b] With $H_0=50\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$, the present critical density becomes $\rho_c=4.7\times10^{-30}\,\mathrm{g\,cm^{-3}}$.

Useful relationships and quantities (Friedmann universe)

Differential volume:

$$\mathrm{d}V=\frac{R_H^3}{(1+z)^3}\frac{\{q_0z+(q_0-1)[(1+2q_0z)^{1/2}-1]\}^2}{q_0^4(1+2q_0z)^{1/2}}\,\mathrm{d}\Omega\,\mathrm{d}z,$$

where

$$R_H=\frac{c}{H_0},\quad\text{the Hubble radius.}$$

Time differential:

$$\mathrm{d}t=-\mathrm{d}z/[(1+z)^2H_0(1+2q_0z)^{1/2}].$$

Look-back time:

$$\tau=-\int_0^z\mathrm{d}t,$$

$q_0=0$,

$$\tau=\frac{1}{H_0}(1-1/(1+z));\quad\lim_{z\to\infty}\tau=1/H_0,$$

$q_0=1/2$,

$$\tau=\frac{2}{3H_0}(1-1/(1+z)^{3/2});\quad\lim_{z\to\infty}\tau=2/3(1/H_0),$$

$q_0=1$,

$$\tau=\frac{1}{H_0}[(2z+1)^{1/2}/(z+1)+2\tan^{-1}(2z+1)^{1/2}-1-\pi/2];$$

$$\lim_{z\to\infty}\tau=0.57(1/H_0),$$

© Cambridge University Press • Provided by the NASA Astrophysics Data System

where

H_0 = Hubble constant,
q_0 = deceleration constant,
z = redshift.

Redshift-magnitude relationship (Friedmann universe):

$$m_{\text{bol}} = 5\log\left\{\frac{c}{H_0 q_0^2}[1 - q_0 + q_0 z + (q_0 - 1)(2q_0 z + 1)^{1/2}]\right\} + M_{\text{bol}} + 25 = 5\log D_{\text{L}} + M_{\text{bol}} + 25.$$

where m_{bol} is the apparent bolometric magnitude of a source with absolute bolometric magnitude M_{bol} and redshift z.
Expanded in powers of z:

$$m_{\text{bol}} = 5\log\left(\frac{zc}{H_0}\right) + 1.086(1 - q_0)z + \cdots + M_{\text{bol}} + 25,$$

where H_0 is in $\text{km s}^{-1}\,\text{Mpc}^{-1}$; zc is in km s^{-1}, and z is the observed redshift.

$$D_{\text{L}} = \frac{c}{H_0 q_0^2}[1 - q_0 + q_0 z + (q_0 - 1)(2q_0 z + 1)^{1/2}]$$

$$\approx \frac{cz}{H_0}[1 + 0.5z(1 - q_0)], \quad q_0 z << 1.$$

$m_{\text{bol}} - M_{\text{bol}} = m - M - K - A$, $m - M$ = observed distance modulus, for heterochromatic magnitudes, where K = redshift correction, A = interstellar absorption.

$$K = 2.5\log(1 + z) + 2.5\log\frac{\int_0^\infty I(\lambda)S(\lambda)\,\mathrm{d}\lambda}{\int_0^\infty I\left(\frac{\lambda}{1+z}\right)S(\lambda)\,\mathrm{d}\lambda}\,[\text{mag}],$$

where $I(\lambda)$ is the incident energy flux per unit wavelength and $S(\lambda)$ is the photometer response function.

Angular diameter–redshift relationship (Friedmann universe)

$$\theta = \frac{l(1 + z)^2}{D_{\text{L}}},$$

where

θ = apparent angular diameter of source,
l = linear diameter of spherical source,
D_{L} = luminosity distance.

© Cambridge University Press • Provided by the NASA Astrophysics Data System

Observed energy flux density (Friedmann universe)

$$S_0(E_0) = \frac{P_e((1+z)E_0)}{4\pi D_L^2},$$

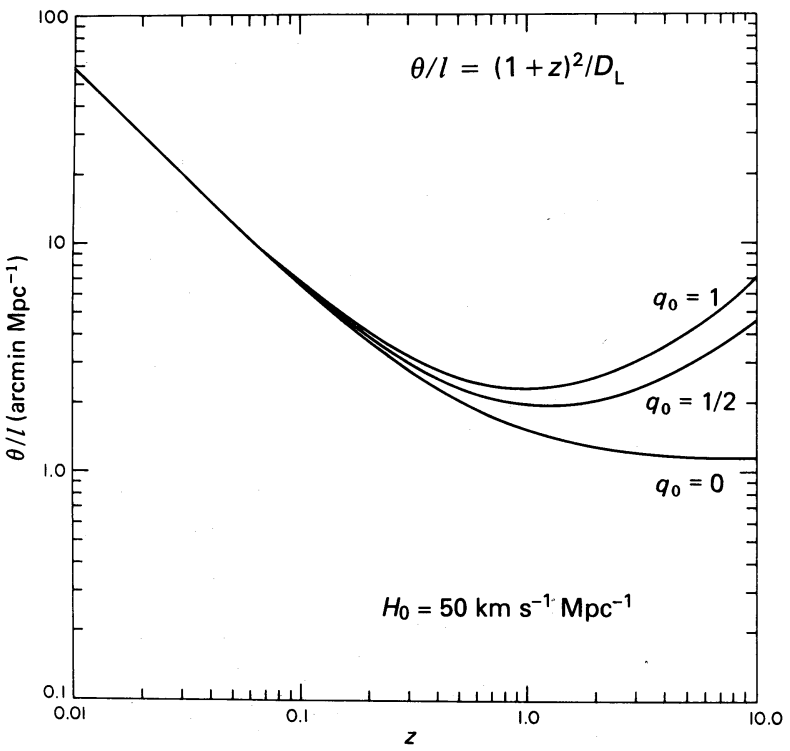
where P_e is the monochromatic power emitted at the energy $(1+z)E_0$, and S_0 is the observed monochromatic energy flux density at the energy E_0. The observed energy flux density integrated from E_1 to E_2 is:

$$F_0(E_1, E_2) = \frac{1}{4\pi D_L^2} \int_{(1+z)E_1}^{(1+z)E_2} p_e(E)\, dE,$$

where $p_e(E)$ is the differential power emitted per unit energy in the emitted rest frame.

Redshift functions

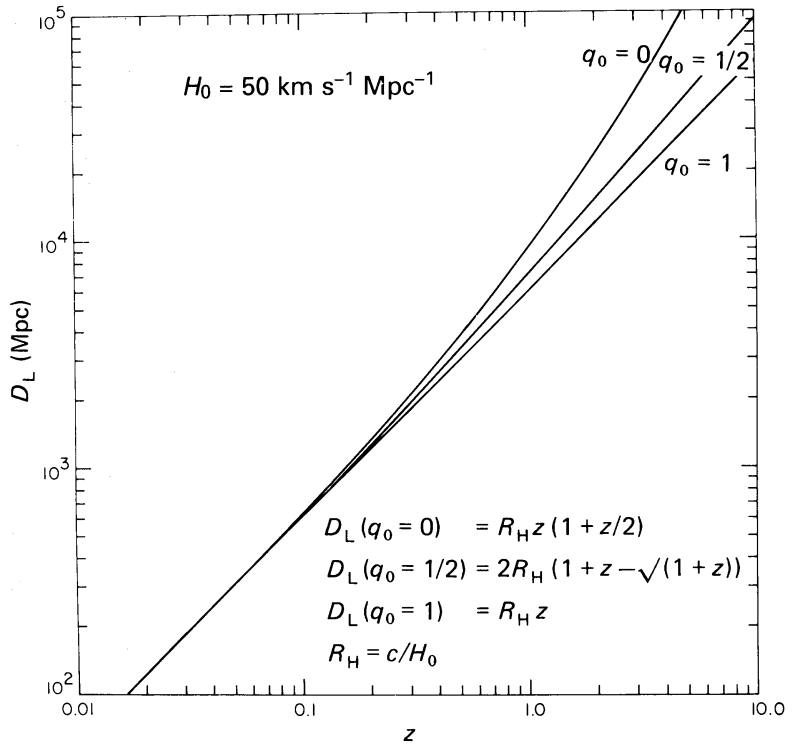
Angular size vs. redshift



© Cambridge University Press • Provided by the NASA Astrophysics Data System

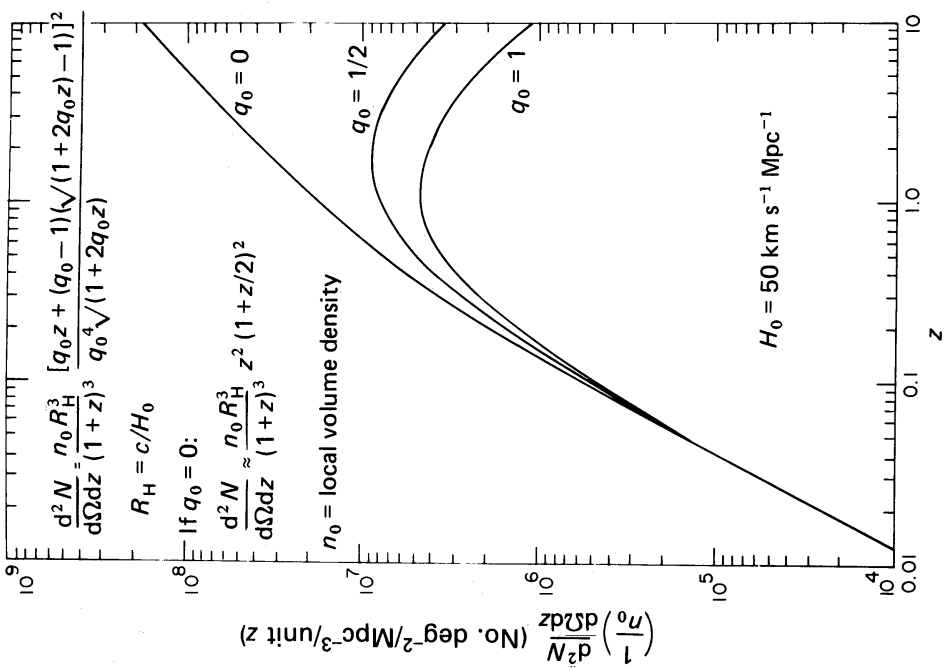
Redshift functions (cont.)

Luminosity distance vs. redshift

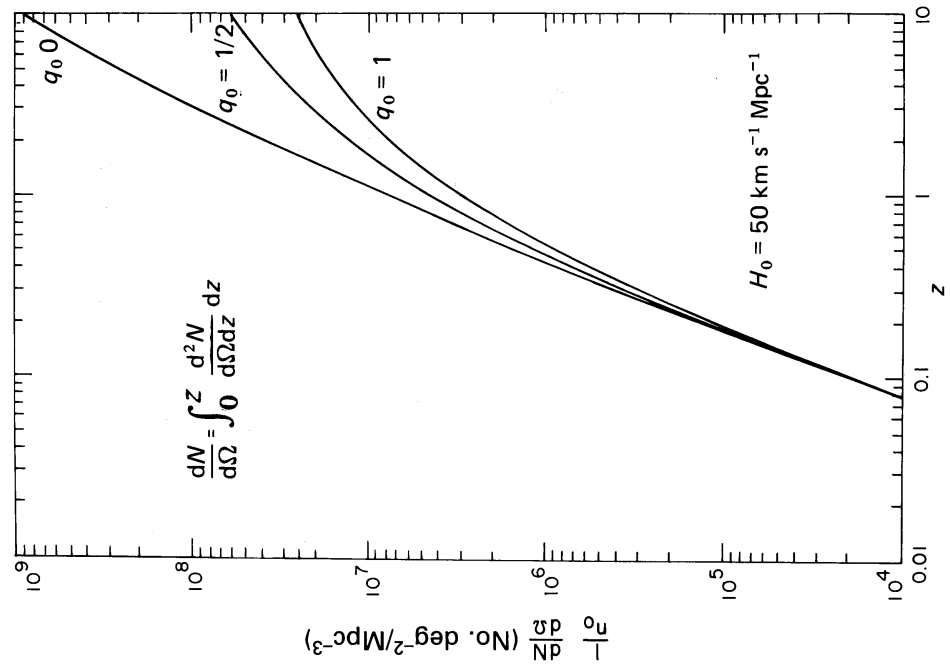


© Cambridge University Press • Provided by the NASA Astrophysics Data System

Angular density of sources per unit z vs. redshift.



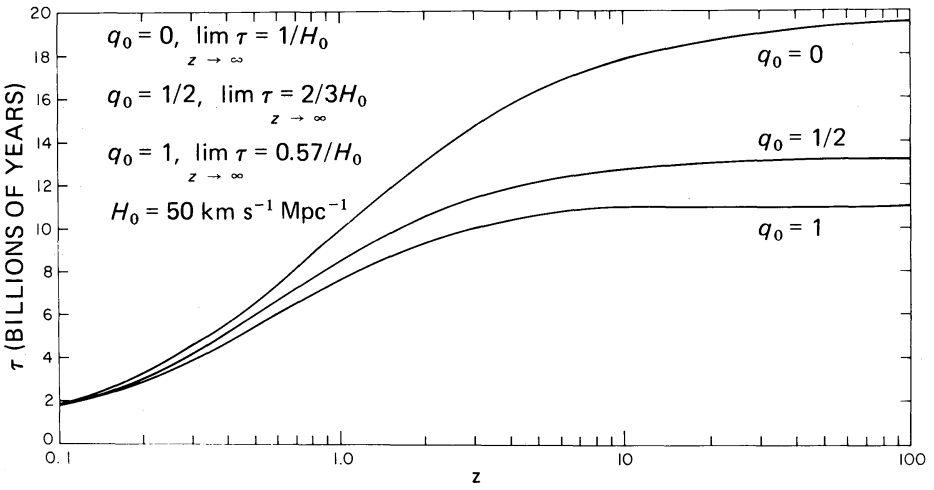
Integrated angular density vs. redshift.



© Cambridge University Press • Provided by the NASA Astrophysics Data System

Redshift functions (cont.)

Look-back time



Bibliography

Gravitational and Cosmology, Weinberg, S., John Wiley and Sons, 1971.

General Relativity and Cosmology, 2nd ed. McVittie, G. C., University of Illinois Press, 1965.

© Cambridge University Press • Provided by the NASA Astrophysics Data System